MAS224, Actuarial Mathematics: Facts from Probability

Handout for Lecture 10 (28/1/02)

Notation:

$P(A)$ = Probability that the event described by A occurs.
$P(A|B)$= Probability that A occurs given that B has occurred (conditional probability)

Note:

(a) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, always.

(c) If A and B are mutually exclusive, i.e. $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

(d) If $A_1 \cap A_2 = \emptyset$ then $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$.

(e) If $A \subset B$ (i.e. A implies B) then $P(A|B) = \dfrac{P(A)}{P(B)}$.

(f) $P(A \cap B) = P(A)P(B)$ **if and only if** the events A and B are independent.

Cumulative *distribution function* (c.d.f.): $F_X(x) = P(X \leq x)$

Two types of random variables:- discrete and continuous

Discrete random variables:

X takes on values from a discrete set $\{x_1, x_2, \ldots\}$. In this case the probability distribution is normally described by the p.m.f. $x_k \mapsto P(X = x_k)$ and $F_X(x) = \sum_{x_k \leq x} P(X{=}x_k)$.

Continuous random varaibles X:

There exists a continuous function $f_X(x)$, called *probability density function* (p.d.f.), such that

(g) $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du.$

The c.d.f. $F_X(x)$ and p.d.f. $f_X(x)$ for a continuous r.v. X are related via $\frac{d}{dx}F_X(x) = f_X(x)$ which holds at every x where the derivative exists.

The p.d.f. is non-negative and integrates to 1, i.e. $\int_{-\infty}^{+\infty} f_X(x)dx = 1$. It is common to show $f_X(x)$ only for those values of x where this function is not zero, i.e. the range of X.

Note:

(h)
$$\begin{aligned} P(X > x) &= 1 - F_X(x);, && \text{(holds always, follows from } P(X > x) + P(X \le x) = 1) \\ &= \sum_{x_k > x} P(X = x_k) && \text{(if } X \text{ is a discrete r.v.)} \\ &= \int_x^{+\infty} f_X(u)du. && \text{(if } X \text{ is a continuous r.v.)} \end{aligned}$$

(i)
$$\begin{aligned} P(x < X \le y) &= F_X(y) - F_X(x); && \text{(holds always)} \\ &= \sum_{x < x_k \le y} P(X = x_k) && \text{(if } X \text{ is a discrete r.v.)} \\ &= \int_x^{y} f_X(u)du. && \text{(if } X \text{ is a continuous r.v.)} \end{aligned}$$

In particular, for continuous X:

$$\begin{aligned} P(x \le X \le x + h) &= F_X(x + h) - F_X(x) \\ &\approx \frac{d}{dx}F_X(x) \times h, \quad \text{for small } h, \\ &= f_X(x)h \end{aligned}$$

The latter relation can be used to give an alternative definition of continuous type random variable. We can say that X is continuous type if there exist a continuous function $f_X(x)$ such that the probabilities $P(x < X \le x + h)$, for all x, can be approximated for small values of h by $f_X(x)h$.

(j) If X is a continuous random variable then

$$P(x_1 < X \le x_2) = P(x_1 \le X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2).$$

Expected Value of X:

$$\begin{aligned} E(X) &= \sum_{x_k} x_k\, P(X = x_k), && \text{if } X \text{ is a discrete random variable;} \\ &= \int_{-\infty}^{+\infty} u\, f_X(u)\, du, && \text{if } X \text{ is a continuous random variable.} \end{aligned}$$

Note: $E(aX + bY) = aE(X) + bE(Y)$.

Variance of X (mean quadratic deviation from $E(X)$):

$$\text{var}(X) = E([X - E(X)]^2) = E(X^2) - [E(X)]^2$$

Note: $\text{var}(aX + bY) = a^2 \text{var}(X) + b^2 \text{var}(Y) + 2ab \,\text{cov}(X, Y)$, where $\text{cov}(X, Y) = E([X - E(X)][Y - E(Y)])$ is the covariance of X and Y. If X and Y are independent then $\text{cov}(X, Y) = 0$ and $\text{var}(aX + bY) = a^2 \text{var}(X) + b^2 \text{var}(Y)$.

Conditioning a random variable X by the event $X > t$.

If X is a continuous random variable and t belongs to the range of its values, then sometimes we need to know the conditional distribution of the random variable $X - t$ given that $X > t$. This conditional distribution can be described by the conditional c.d.f.

$$\begin{aligned} F_{(X-t)|(X>t)}(s) &\stackrel{\text{def.}}{=} P(X - t \leq s | X > t) \\ &= P(t < X \leq t + s | X > t) \end{aligned}$$

To follow the last equality, note that the event that $X - t \leq s$ is the same as $X \leq t + s$, and also, the conditional probability of $X \leq t$ given that $X > t$ is zero, i.e. $P(X \leq t | X > t) = 0$. Thus, by property (d),

$$P(X-t \leq s|X>t) = P(X \leq t|X>t) + P(t<X \leq t+s|X>t) = P(t<X \leq t+s|X>t).$$

The conditional probability density function $f_{(X-t)|(X>t)}$ of the random variable $X - t$ conditioned by the event $X > t$ can be found by calculating $P(s < X - t \leq s + h \mid X > t)$ for small h and non-negative s:

$$\begin{aligned} f_{(X-t)|(X>t)}(s) \times h &\approx P(s < (X - t) \leq s + h \mid X > t) && \text{[by (i)]} \\ &= \frac{P(s < X - t \leq s + h)}{P(X > t)} && \text{[by (a) and (e)]} \\ &= \frac{P(s + t < X \leq s + t + h)}{P(X > t)} \\ &\approx \frac{f_X(s + t) \times h}{P(X > t)}, && \text{[by (i)]} \end{aligned}$$

Obviously $f_{(X-t)|(X>t)}(s) = 0$ for negative s. Indeed, if $s < 0$ then the events $s < (X-t) \leq s+h$ and $X > t$ are disjoint for all sufficiently small h and $P(s < (X - t) \leq s + h \mid X > t) = 0$.

Therefore

(k) $$f_{(X-t)|(X>t)}(s) = \frac{f_X(s + t)}{P(X > t)} = \frac{f_X(s + t)}{1 - F_X(t)} \text{ if } s \geq 0 \text{ and } f_{(X-t)|(X>t)}(s) = 0 \text{ if } s < 0.$$

The conditional expectation of $X - t$ given $X > t$ is denoted by $E(X - t|X > t)$. This is the expected value of $X - t$ with respect to the conditional distribution given by the conditional p.d.f. $f_{(X-t)|(X>t)}(s)$:

(l) $$E(X - t|X > t) = \int_0^\infty s\, f_{(X-t)|(X>t)}(s)\, ds = \frac{\int_0^\infty s\, f_X(s + t)\, ds}{P(X > t)} = \frac{\int_t^\infty (u - t)\, f_X(u)\, du}{P(X > t)}.$$

Examples of distributions:

1. Bernoulli distribution (discrete-type): $X \sim$ Bernoulli(p),

   X takes on either 1 or 0 (success or failure); $P(X=1) = p$, $P(X=0) = q$; $p+q = 1$; $E(X) = p$, $\text{var}(X) = p\,q$

2. Binomial distribution (discrete-type): $X \sim$ Bin(n, p),

   X takes on the values $0, 1, 2, \ldots n$; $P(X=k) = C_k^n p^k q^{n-k}$, $k = 0, 1, \ldots, n$; $p+q = 1$; $E(X) = np$, $\text{var}(X) = np\,q$

   If $X_1, X_2, \ldots X_n$ are mutually independent and $X_j \sim$ Bernoulli(p) for all j then

   $$X_1 + X_2 + \ldots X_n \sim \text{Bin}(n, p).$$

3. Uniform distribution on $[0, 1]$ (continuous-type): $X \sim$ Uniform$[0, 1]$,

   X can take on any value between 0 and 1; $P(a \leq X \leq b) = b - a$ for any $0 \leq a < b \leq 1$.
   p.d.f.: $f_X(x) = 1$ if $x \in [0, 1]$ and $f_X(x) = 0$ otherwise.
   $E(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{3}$

4. Exponential distribution (continuous-type): $X \sim$ Exp(λ),

   X can take on any non-negative value;
   p.d.f.: $f_X(x) = \lambda e^{-\lambda x}$ if $x \geq 0$ and $f_X(x) = 0$ otherwise
   c.d.f.: $F_X(x) = 1 - e^{-\lambda x}$ if $x \geq 0$ and $F_X(x) = 0$ otherwise
   $E(X) = \frac{1}{\lambda}$, $\text{var}(X) = \frac{1}{\lambda^2}$

   If $X \sim$ Exp(λ) then the conditional distribution of $X - t$ given that $X > t$ is also Exp(λ). Indeed,

   $$\begin{aligned} f_{(X-t)|(X>t)}(s) &= \frac{f_X(s+t)}{P(X>t)} = \frac{f_X(s+t)}{1-F_X(t)} = \frac{\lambda e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= \lambda e^{-\lambda s} = f_X(s) \qquad\qquad s, t \geq 0. \end{aligned}$$

   Also, if $X \sim$ Exp(λ) then $P(X - t > s | X > t) = P(X > s)$.